Evaluating the Emittance Increase Due to the RF Coupler Fields

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Abstract

This technical note proposes a method for evaluating the emittance due to the time-dependent kick of the RF couplers. The evaluation assumes the 3D coupler fields are known from an EM code calculation and uses these fields to compute the emittance they produce. This note considers the field components which give the beam a time-dependent dipole kick. Expressions are given for the phase emittances of the transverse dipole field kicks and for the longitudinal magnetic field of the coupler.

Introduction

Other approaches for computing the coupler emittance have been based upon the RF properties of the coupler and have provided useful estimates of the coupler fields and their effects on the beam quality. See for example, M. Dohlus et al. and V. Shemelin et al. In this report the 3D field map produced by a design code such as Microwave Studio, etc. is integrated in the equations of motion with some simplifying assumptions to give expressions for the emittance due to the time-dependence of the coupler fields.

The Phase Dependent Emittance due to the Transverse Dipole Fields

The derivation begins with the 1-D relativistic equation of motion

$$\frac{d}{dt}(\gamma m\dot{x}) = e\left(E_x + \dot{y}B_z - \dot{z}B_y\right) \tag{1}$$

Since the beam is traveling with constant velocity along the z-axis, then $\dot{z} = \beta c$ and $\dot{y} = \beta c y'$ and the equation of motion becomes the paraxial ray equation,

$$\gamma\beta^2 mc^2 \frac{d}{dz}x' = e\left\{E_x + \beta c\left(y'B_z - B_y\right)\right\} \tag{2}$$

Next assume the beam is rigid, that is, its transverse size is constant while it's in the coupler fields. Therefore $y' = 0$ and the B_z term can be dropped. Also since the bunch is travelling at constant velocity (for now ignore the longitudinal E-fields), time in the rf fields can be replaced with $\frac{z}{\beta c}$. The RF fields are assumed to be

$$\vec{E}(x,y,z,t) = \vec{E}(x,y,z)sin(\omega t + \phi)$$

and

$$\vec{B}(x,y,z,t) = \vec{B}(x,y,z)\cos(\omega t + \phi)$$

Making these changes and integrating Eq. (2) gives the transverse kick in the x-plane,

$$\int \frac{d}{dz}x'dz = \Delta x'(x,y,\phi) = \frac{e}{\gamma\beta^2 mc^2}\left[\int E_x(x,y,z)sin\left(\frac{2\pi}{\beta\lambda}z + \phi\right)dz - \beta c\int B_y(x,y,z)cos\left(\frac{2\pi}{\beta\lambda}z + \phi\right)dz\right] \tag{3}$$

Here λ is the rf wavelength and ϕ is the rf phase. In the analysis described below in the summary section, these integrals should be done using the 3D field maps generated by the engineering design codes such as ANSYS, Microwave studio, etc. used to design the coupler.

Computing these integrals along the z-axis and over a range of (x,y) coordinates will quantify how much the couplers degrade the beam quality and establish the usable aperture. The summary section describes how the integrals around a circle in the (x,y) plane can be used to obtain the dipole and quadrupole fields.

Given that the transverse coupler fields are rather smooth, single-valued functions they can be characterized by their peak value and an effective length. The effective length for the transverse electric field E_x is defined to be

$$L_{E_x} = \frac{\int E_x(x=0,y=0,z)dz}{E_x^{peak}} \tag{4}$$

And similarly for the effective length of the transverse magnetic field,

$$L_{B_y} = \frac{\int B_y(x=0,y=0,z)dz}{B_y^{peak}} \tag{5}$$

In this effective length model the field is represented by a constant peak value over the effective length. Following this condition, the integrals of Eqn. (3) can be written as

$$\Delta x' = \frac{e}{\gamma\beta^2 mc^2}\left\{E_x^{peak}\int_0^{L_{E_x}} sin\left(\frac{2\pi}{\beta\lambda}z+\phi\right)dz - \beta c B_y^{peak}\int_0^{L_{B_y}} cos\left(\frac{2\pi}{\beta\lambda}z+\phi\right)dz\right\} \tag{6}$$

Performing the integrals gives the x-angle kick due to the coupler E_x and B_y fields characterized in terms of their effective lengths and peak fields,

$$\Delta x'(x,y,\phi) = \frac{e}{\beta\gamma mc^2}\frac{\lambda}{2\pi}\left\{E_x^{peak}\left[cos\phi - cos\left(\phi+\frac{2\pi}{\beta\lambda}L_{E_x}\right)\right] - \beta c B_y^{peak}\left[sin\left(\phi+\frac{2\pi}{\beta\lambda}L_{B_y}\right)-\sin\phi\right]\right\} \tag{7}$$

Another approach expresses the kick in terms of the beam's transverse voltage gain, V_x, due to the coupler fields. In this case, the angle kick is related to a transverse voltage as,

$$\Delta x'(x,y,\phi) = \frac{e}{\beta\gamma mc^2}V_x(x,y,\phi). \tag{9}$$

The transverse voltage gain can be written as a complex quantity in which the real part is given by the electric field

$$Re\, V_x(x,y,\phi) = \int E_x(x,y,z)\, sin\left(\frac{2\pi}{\beta\lambda}z+\phi\right)dz = \frac{\lambda}{2\pi}E_x^{peak}\left[cos\phi - cos\left(\phi+\frac{2\pi}{\beta\lambda}L_{E_x}\right)\right] \tag{10}$$

while the imaginary voltage is due to the magnetic field is

$$Im\, V_x(x,y,\phi) = -\int \beta c B_y(x,y,z) cos\left(\frac{2\pi}{\beta\lambda}z+\phi\right)dz = -\frac{\lambda}{2\pi}\beta c B_y^{peak}\left[sin\left(\phi+\frac{2\pi}{\beta\lambda}L_{B_y}\right)-\sin\phi\right] \tag{11}$$

Similar expressions can be written for the y-plane kicks.

A reasonable definition for the dipole-phase emittance due to the dipole kick is for a initially collimated (zero divergence) beam with a transverse rms size of σ_x and a bunch rms phase length or phase spread of σ_ϕ is

$$\epsilon_{x-dipole,\phi}(x,y,\phi) = \beta\gamma\sigma_x\sigma_\phi\left|\frac{d}{d\phi}\Delta x'(x,y,\phi)\right| = \sigma_x\sigma_\phi\frac{e}{mc^2}\left|\frac{d}{d\phi}V_x(x,y,\phi)\right| \tag{12}$$

Notice that the normalized emittance doesn't depend upon the beam energy, but only upon the transverse voltage gain. This means an electron transiting a coupler at a higher energy gains the same transverse voltage as a lower energy electron, and therefore gains the same emittance increase. The coupler emittance gain at higher beam energy is lower because the beam size is smaller at higher energy due to effects such as Landau damping.

Computing the derivative of the dipole kick with respect to the rf phase gives the normalized, x-plane dipole-phase emittance as

$$\epsilon_{x-dipole} = \sigma_x \sigma_\phi \frac{\beta\lambda}{2\pi} \frac{e}{mc^2} \left| E_x^{peak} \left(sin\left(\phi + \frac{2\pi}{\beta\lambda} L_{E_x}\right) - \sin\phi \right) - \beta c B_y^{peak} \left(cos\left(\phi + \frac{2\pi}{\beta\lambda} L_{B_y}\right) - \cos\phi \right) \right| \quad (13)$$

Here the peak fields and effective lengths are evaluated along the z-axis at x=0 and y=0.

Emittance due to the Longitudinal B-Field

The above discussion describes the phase emittance due to the dipole kicks of the coupler transverse electric and magnetic fields. In addition to these angle kicks, there is the phase emittance produced by the longitudinal magnetic field, B_z, as well as the emittance of the longitudinal electric field. Modeling a longitudinal E-field would require including the acceleration term, γ', in the equation of motion. In this report the longitudinal E-field is ignored and the phase emittance of the Bz-field is derived.

The phase emittance of an rf lens produced by a Bz-field can be written as

$$\epsilon_{B_z,\phi} = \beta\gamma\sigma_x^2\sigma_\phi \left| \frac{d}{d\phi}\left(\frac{1}{f_{B_z}}\right) \right| \quad (14)$$

Where f_{B_z} is the focal length of the B_z lens which is given by

$$\frac{1}{f_{B_z}} = \left(\frac{e}{2\beta\gamma mc}\right)^2 \int \left[B_z(x,y,z) sin\left(\frac{2\pi}{\beta\lambda} z + \phi\right)\right]^2 dz \quad (15)$$

This expression is similar to those given above for the dipole kicks and can also be written in terms of a peak field and effective length.

Taking the derivative of Eqn. (15) wrt the phase and using it in Eqn. (14) gives the phase emittance of time-varying B_z-field,

$$\epsilon_{B_z,\phi}(x,y,\phi) = \frac{1}{\beta\gamma}\left(\frac{e}{2mc}\right)^2 \sigma_x^2\sigma_\phi \int B_z(x,y,z)^2 sin\left[2\left(\frac{2\pi}{\beta\lambda} z + \phi\right)\right] dz \quad (16)$$

Using the effective length model for the Bz-field, and assuming the integrals for B_z^2 and *sin* are separable the emittance becomes

$$\epsilon_{B_z,\phi} = \left(\frac{e}{2mc}\right)^2 \sigma_x^2\sigma_\phi \left(B_z^{peak}\right)^2 \frac{\lambda}{4\pi\gamma} \left[\cos 2\phi - \cos\left(\frac{4\pi}{\beta\lambda} L_{B_z} + 2\phi\right)\right] \quad (17)$$

Again as for the dipole kicks, the 3D field maps provide $B_z(x,y,z)$ which is integrated along lines of constant (x,y) to produce a 2D map of the phase emittance. This emittance should be added in quadrature with the dipole kick emittances for the total emittance.

Phase Emittance due to Quadrupole Fields

For an electron travelling in electric and magnetic quadrupole fields the x- and y-plane equations of motion are (*Theory and Design of Charged Particle Beams* by M. Reiser, p.113),

$$x'' + \kappa x = 0 \quad (18)$$

$$y'' - \kappa y = 0 \quad (19)$$

Where κ depends upon whether the field is electric,

$$\kappa_{E_x} = \frac{e}{\beta^2\gamma mc^2}\left(\frac{E_x}{a}\right)_0 sin(\omega t+\phi) \tag{20}$$

or magnetic,

$$\kappa_{B_y} = \frac{e}{\beta\gamma mc}\left(\frac{B_y}{a}\right)_0 cos(\omega t+\phi) \tag{21}$$

The quantity $\left(\frac{E_x}{a}\right)_0$ is the gradient of the E_x field over a distance a along the x-direction near the nominal center at x=0, y=0. And $\left(\frac{B_y}{a}\right)_0$ is the B_y gradient in the x-direction. Details of how to determine the gradients from the field map is described in the procedure section.

Adding the forces on an electron by the coupler's electric and magnetic field gradients in the x-plane equation of motion gives

$$x'' + \frac{e}{\beta^2\gamma mc^2}\left[\left(\frac{E_x}{a}\right)_0 sin(\omega t+\phi) - \beta c\left(\frac{B_y}{a}\right)_0 cos(\omega t+\phi)\right]\ x = 0 \tag{22}$$

Again, since $t = \frac{z}{\beta c}$ and $\omega = \frac{2\pi c}{\lambda}$, this expression can be written as

$$x'' + \frac{e}{\beta^2\gamma mc^2}\left[\left(\frac{E_x}{a}\right)_0 sin\left(\frac{2\pi}{\beta\lambda}z+\phi\right) - \beta c\left(\frac{B_y}{a}\right)_0 cos\left(\frac{2\pi}{\beta\lambda}z+\phi\right)\right]\ x = 0 \tag{23}$$

Integrating gives the kick of the quadrupole fields,

$$\Delta x' = -x\,\frac{e}{\beta^2\gamma mc^2}\left[\int_0^{L_{E_x}}\left(\frac{E_x}{a}\right)_0 sin\left(\frac{2\pi}{\beta\lambda}z+\phi\right)dz\ -\beta c\int_0^{L_{B_y}}\left(\frac{B_y}{a}\right)_0 cos\left(\frac{2\pi}{\beta\lambda}z+\phi\right)dz\right] \tag{24}$$

Similar to the previous discussion on the dipole fields, it is assumed the beam is rigid with constant the transverse size while it's in the coupler fields. In the emittance calculation the initial beam divergence is assumed to be zero. In addition an effective length model for the fields is applied which constrains the integration limits to $0 < z < L_{E_x}$ and $0 < z < L_{B_y}$ over which the field gradients have constant values of $\left(\frac{E_x}{a}\right)_0$ and $\left(\frac{B_y}{a}\right)_0$ for the electric and magnetic fields, respectively. With these assumptions the quadrupole kick is found to be

$$\Delta x' = x\,\frac{e}{\beta\gamma mc^2}\,\frac{\lambda}{2\pi}\left[\left(\frac{E_x}{a}\right)_0\left(cos\phi - cos\left(\frac{2\pi}{\beta\lambda}L_{E_x}+\phi\right)\right) - \beta c\left(\frac{B_y}{a}\right)_0\left(sin\left(\frac{2\pi}{\beta\lambda}L_{B_y}+\phi\right) - \sin\phi\right)\right] \tag{25}$$

Since the kick angle has a linear dependence upon the beam's displacement, x, the field behaves as a lens,

$$\Delta x' = -\frac{1}{f}\,x \tag{26}$$

Therefore the electric and magnetic field gradients in the x-direction focus the beam with an optical power of

$$\frac{1}{f_{x-quad}} = -\frac{e}{\beta\gamma mc^2}\,\frac{\lambda}{2\pi}\left[\left(\frac{E_x}{a}\right)_0\left(\cos\phi - cos\left(\frac{2\pi}{\beta\lambda}L_{E_x}+\phi\right)\right) - \beta c\left(\frac{B_y}{a}\right)_0\left(sin\left(\frac{2\pi}{\beta\lambda}L_{B_y}+\phi\right) - \sin\phi\right)\right] \tag{27}$$

Following the assumptions used earlier to obtain the dipole-phase emittance, the quadrupole-phase emittance or the emittance due to the time dependent lensing of the field gradients can be written as

$$\epsilon_{q,\phi} = \beta\gamma\sigma_x^2\sigma_\phi \left|\frac{d}{d\phi}\left(\frac{1}{f}\right)\right| \tag{28}$$

Inserting the derivative of the optical power gives

$$\epsilon_{x-quad,\phi} = \sigma_x^2\sigma_\phi \frac{\lambda e}{2\pi mc^2}\left|\left(\frac{E_x}{a}\right)_0\left(\cos\phi - cos\left(\frac{2\pi}{\beta\lambda}L_{E_x}+\phi\right)\right) - \beta c\left(\frac{B_y}{a}\right)_0\left(sin\left(\frac{2\pi}{\beta\lambda}L_{B_y}+\phi\right)-\sin\phi\right)\right| \tag{29}$$

Summary of Coupler Field Analysis Procedure

Beginning with high-resolution, 3D field maps provided by the design codes such as microwave studio, etc. various line integrals are taken though the 3D maps. The field maps are assumed to be in Cartesian coordinates with the z-axis being the beam axis. Specifically these codes provide the electric and magnetic field maps, $\vec{E}(x,y,z)$ and $\vec{B}(x,y,z)$ with sufficient resolution to determine the dipole and quadrupole field components. This resolution will require a numerical grid with approximately 0.1 mm spacing. And since the beam can be large, the coupler field effects need to be quantified out to a minimum radius of 5 mm.

In the effective edge model the coupler fields are characterized by 24 parameters, 12 each of the electric and magnetic fields:

$$E_x^{peak}, L_{E_x}, E_{x,0}, \left(\frac{E_x}{a}\right)_0, \theta_{Ex,0};\quad E_y^{peak}, L_{E_y}, E_{y,0}, \left(\frac{E_y}{a}\right)_0, \theta_{Ey,0};\quad E_z^{peak}, L_{E_z}$$

$$B_x^{peak}, L_{B_x}, B_{x,0}, \left(\frac{B_x}{a}\right)_0, \theta_{Bx,0};\quad B_y^{peak}, L_{B_y}, B_{y,0}, \left(\frac{B_y}{a}\right)_0, \theta_{By,0};\quad B_z^{peak}, L_{B_z}$$

In the field analysis these parameters are extracted from the 3D Cartesian field maps, and then used in expressions derived above to estimate the various emittance effects of the coupler fields.

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I wish to thank Houjun Qian (LBNL) for his careful critique of the original tech note and for pointing out a critical error which is corrected in this revision. He correctly pointed out that my analysis of the 3D simulation fields extracted the monopole instead of the much smaller quadrupole field and therefore greatly overestimated the coupler emittance growth. The monopole field is indeed large but it is the linear part of the RF lensing of the SRF linac and for short bunches results in little emittance growth for a properly matched beam. The quadrupole field integrals needed for the above expressions are best obtained from the numerical analysis of Z. Li in the ARDB department at SLAC or using a similar analysis.

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