Calculation of wakefields for plasma-wakefield accelerators

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ICFA mini-Workshop on Impedances and Beam Instabilities in Particle Accelerators

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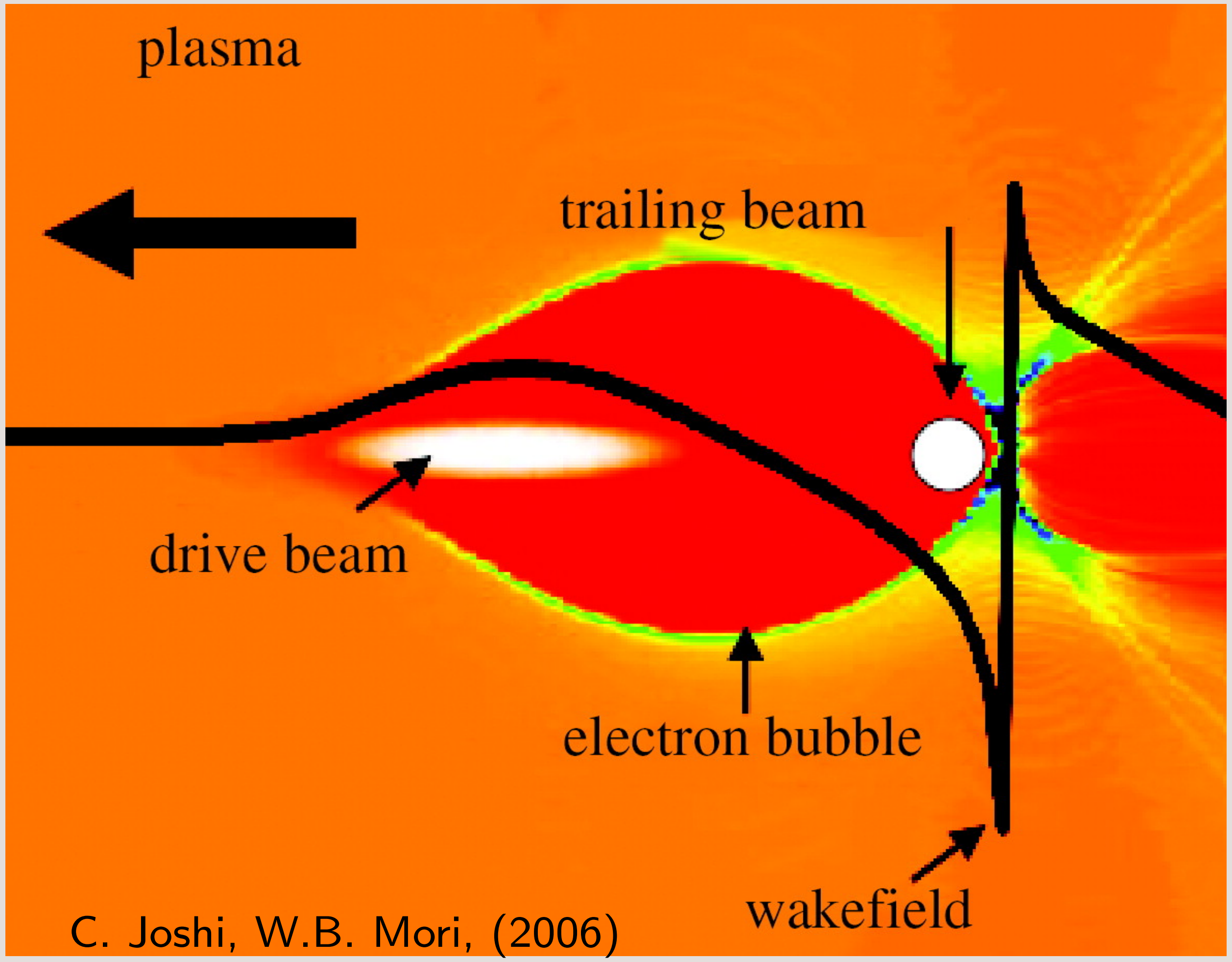
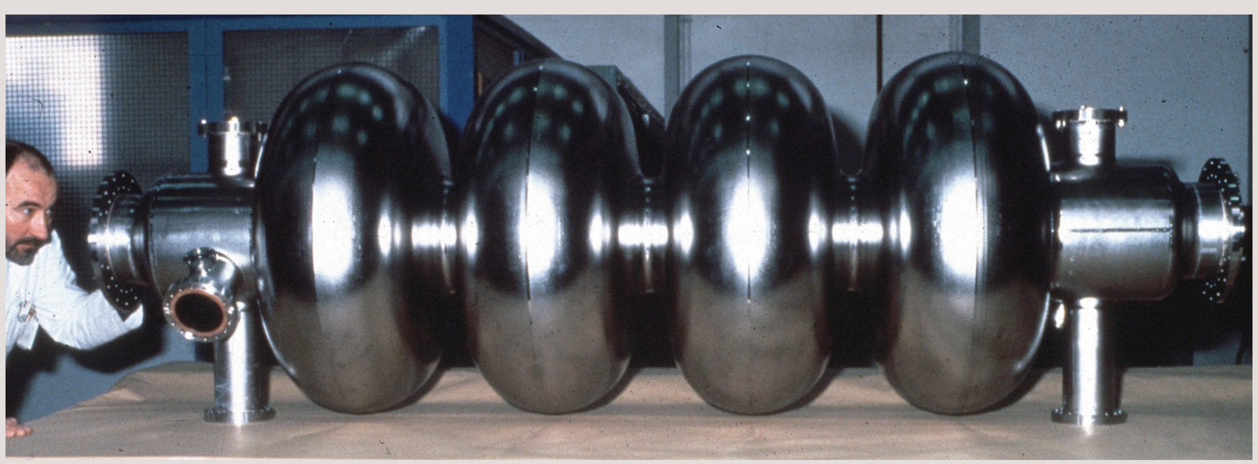
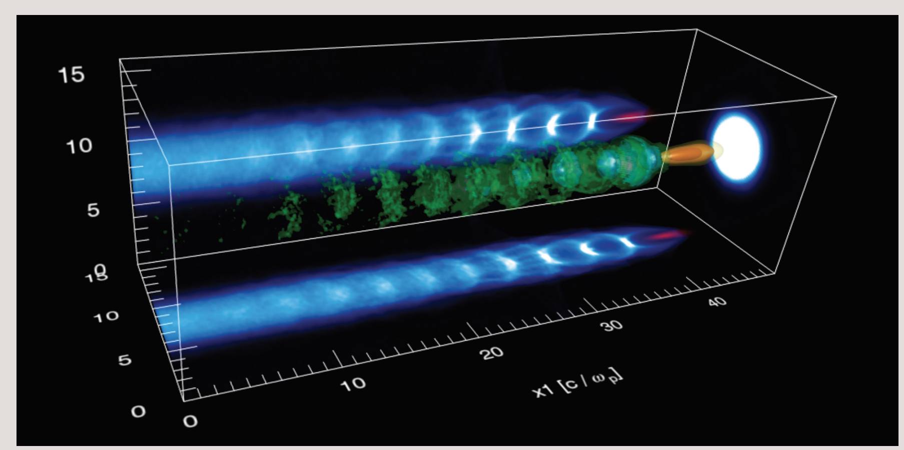
Introduction to PWFA



C. Joshi, W.B. Mori, (2006)

- Plasma wake excited by relativistic particle bunch
- "Blow-out" regime when $n_b/n_e > 1$
- Acceleration and focusing by plasma
- Accelerating field scales as $n_e^{1/2}$
- Typical: $n_e \sim 10^{17}$ cm^{-3}, $k_p^{-1} = 17$ µm, $E \gtrsim 10$ GV/m, $G \gtrsim$ MT/m





1 m, 20 MV/m

100 µm, 20 GV/m

Hosing instability in PWFA

Courtesy of Weiming An from UCLA.

Plasma wakefields

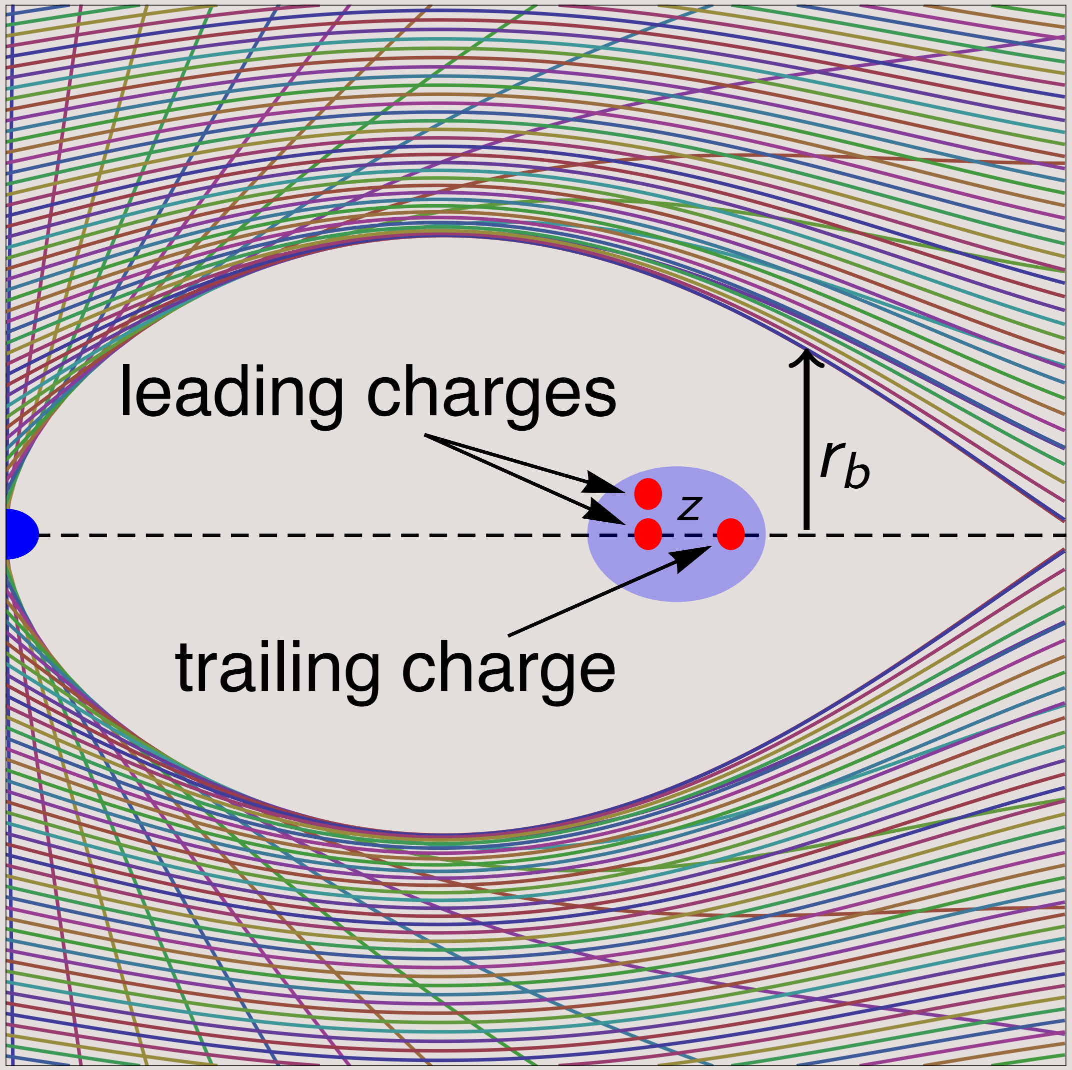
The terminology of wakefields in plasma can be confusing. The original meaning of the wake in plasma is the field generated by the *driver* that accelerates the *witness* beam. The driver is a beam of charged particles (PWFA) or a laser beam (LWFA).

In this presentation, by wakefields I mean the fields (longitudinal and transverse) with which the *witness bunch* acts on itself. They are generated by the *leading* charges and act on the *trailing* charges of the witness bunch.

In *linear approximation*, valid for $n_b \ll n_p$, one can assume that the perturbation of the plasma density is small, $\delta n_e \ll n_e$. The wakefield problem can be solved analytically for arbitrary charge distribution of the driver and witness bunches[1]. This approach, unfortunately, does not work in the blowout regime.

[1] T. Katsouleas et al., Particle Accelerators, **22**, 81 (1987).

Wakefields in the blowout regime



In the absence of theory some researchers[2] use for the *short-range wakefields* formulas that work for a round pipe with resistive wall, corrugated pipe, dielectric pipe, etc. They replace the pipe radius a in these formulas by the bubble radius r_b at the location of the source charge,

$$w_\ell(z) = \frac{4}{r_b^2} h(z) \qquad w_t(z) = \frac{8z}{r_b^4} h(z)$$

$h(z)$ is the step function (in SI system of units multiply by $Z_0 c/4\pi$).

Our goal is to calculate the wakes by solving Maxwell equations with correct plasma responce.

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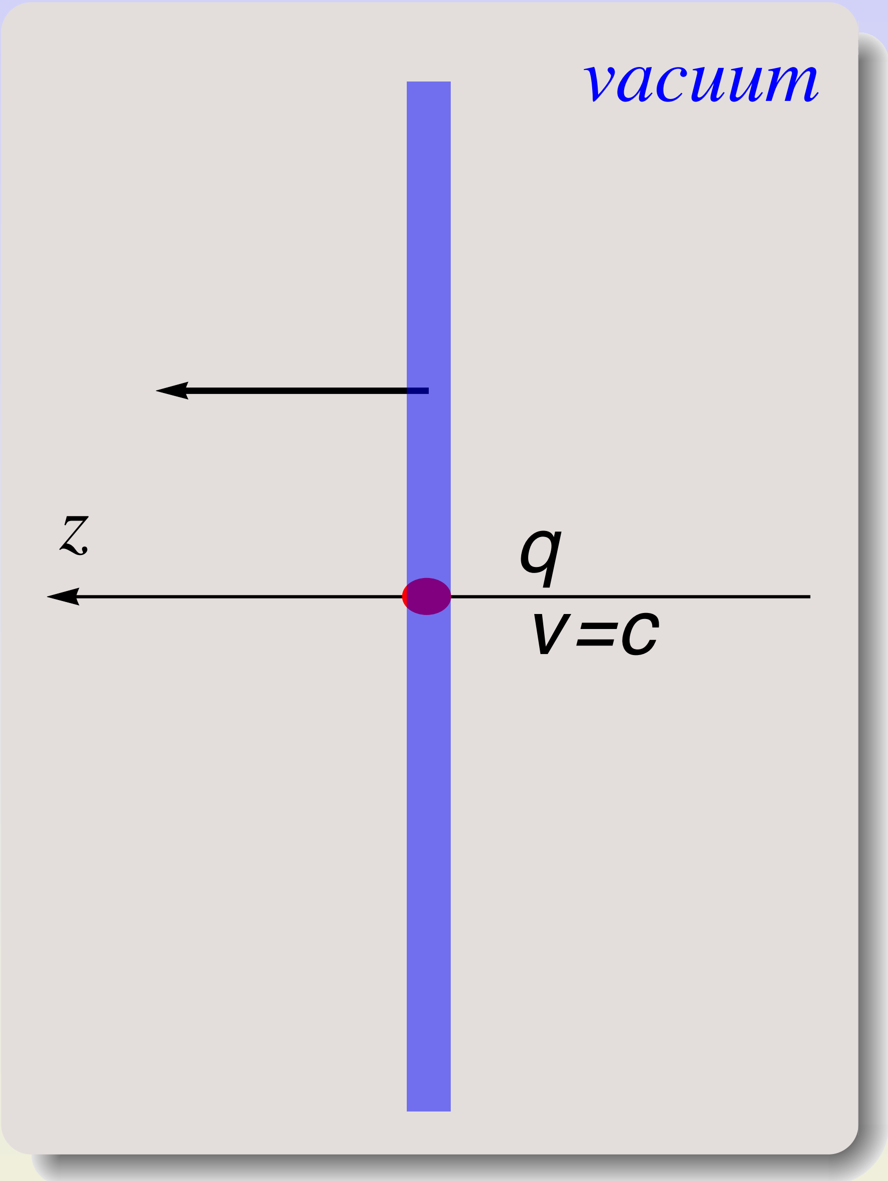
[2] V. Lebedev, A. Burov, S. Nagaitsev, arXiv:1701.01498 (2017).

Relativistic point charge moving in free space

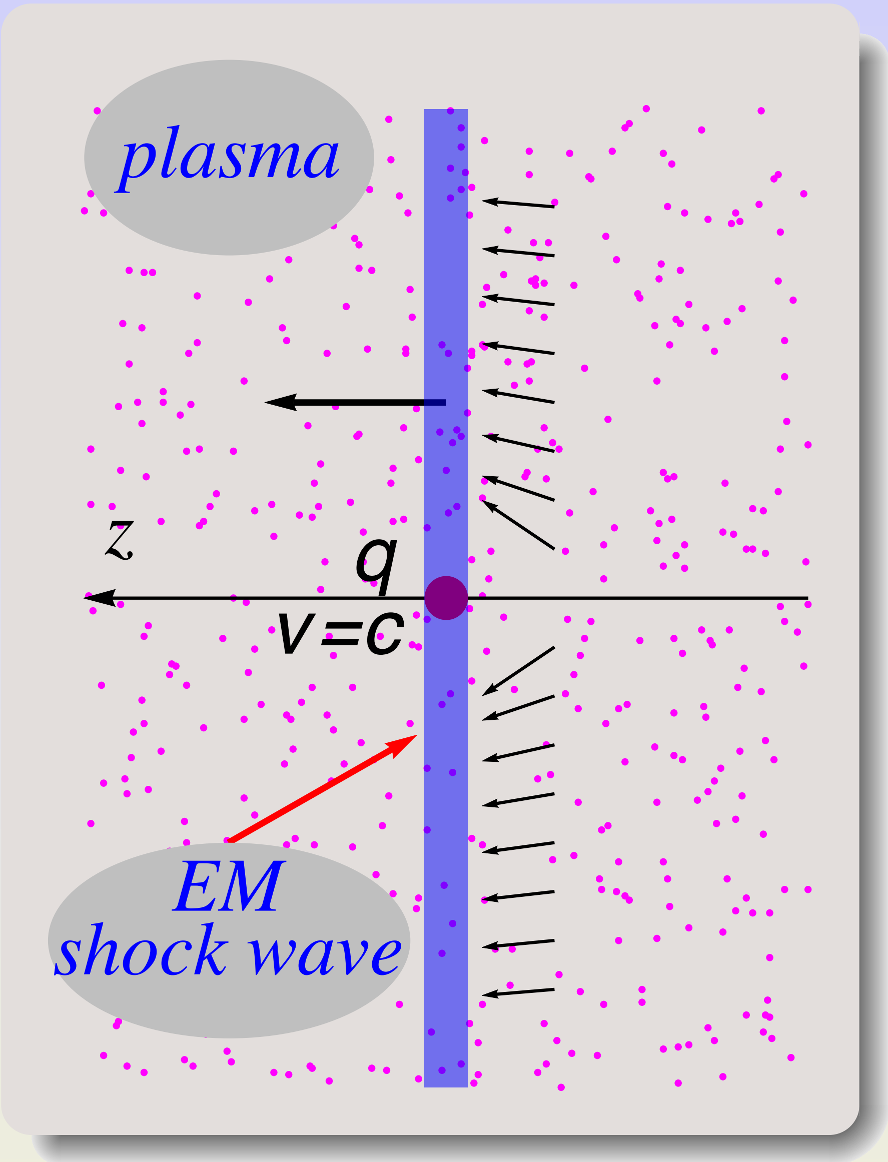


In wakefield theory for relativistic beams we assume $v = c$. When a point charge q is moving in vacuum, its field is

$$E_r = B_\theta = \frac{2q}{r}\delta(z - ct)$$

What happens if the point charge is moving in uniform, cold plasma of density n_0?

Point charge moving through plasma



The remarkable result of Ref.[3] is the existence of the *electromagnetic shock wave* (EMSW)

$$E_r = B_\theta = 2qk_pK_1(k_pr)\delta(z-ct)$$

where $k_p = \omega_p/c = \sqrt{4\pi n_0 e^2/mc^2}$ and K_1 is the modified Bessel function. For $r \ll k_p^{-1}$ we recover $E_r, B_\theta \approx 2q\delta(z-ct)/r$; for $r \gg k_p^{-1}$ the field decays exponentially, $E_r, B_\theta \propto e^{-k_pr}/\sqrt{k_pr}$. Remarkably, the fields in EMSW are linear functions of charge.

The only external dimensionless parameter in the problem is

$$\nu = \frac{q}{e} r_e k_p = N_d r_e k_p \sim q\sqrt{n_0}$$

For $n_0 = 10^{16}$ cm^{-3}, $q = 1$ nC we have $k_p^{-1} = 53$ µm, $\nu = 0.3$.

[3] N. Barov et al., PRAB **7**, 061301 (2004).

Plasma equations

This is the system of equations (in dimensionless units) that governs the plasma dynamics in *axisymmetric* geometry. We assume a steady state with everything depending on $\xi = t - z$ and $r = \sqrt{x^2 + y^2}$. Introduce $\psi = \phi - A_z$,

$$E_z = \partial_\xi \psi, \qquad E_r = -\partial_r \psi$$

$\xi \rightarrow \xi k_p^{-1}$

$E \rightarrow Emc\omega_p/e$

Eq. for ψ

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\psi = n_e(1 - v_z) - 1$$

Eq. for B_θ

$$\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = -\frac{\partial}{\partial \xi} n_e v_r - \frac{\partial}{\partial r} n_e v_z - \frac{\partial n_d}{\partial r} - \frac{\partial n_w}{\partial r}$$

Eqs. of motion for plasma electrons

$$\frac{dp_r}{d\xi} = \frac{\gamma}{1+\psi}\partial_r \psi - B_\theta, \qquad \frac{dr}{d\xi} = \frac{p_r}{1+\psi}, \qquad 1 - v_z = \frac{1}{\gamma}(1+\psi)$$

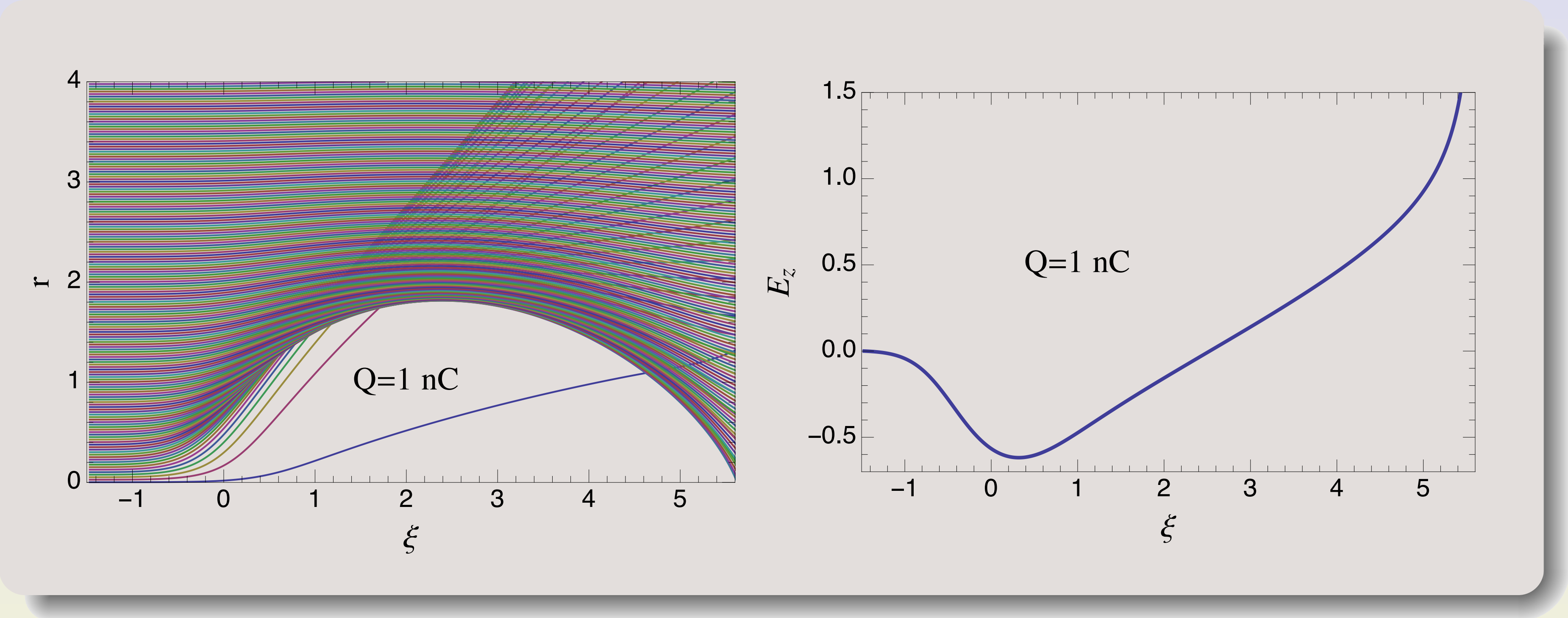
The continuity equation

$$\partial_\xi [n_e(1 - v_z)] + \frac{\partial}{\partial r} r n_e v_r = 0$$

Remarkably, for a given plasma flow, n_e, v_r and v_z, the fields are found through an integration over r in each slice ξ.

Numerical solution of PWFA equations

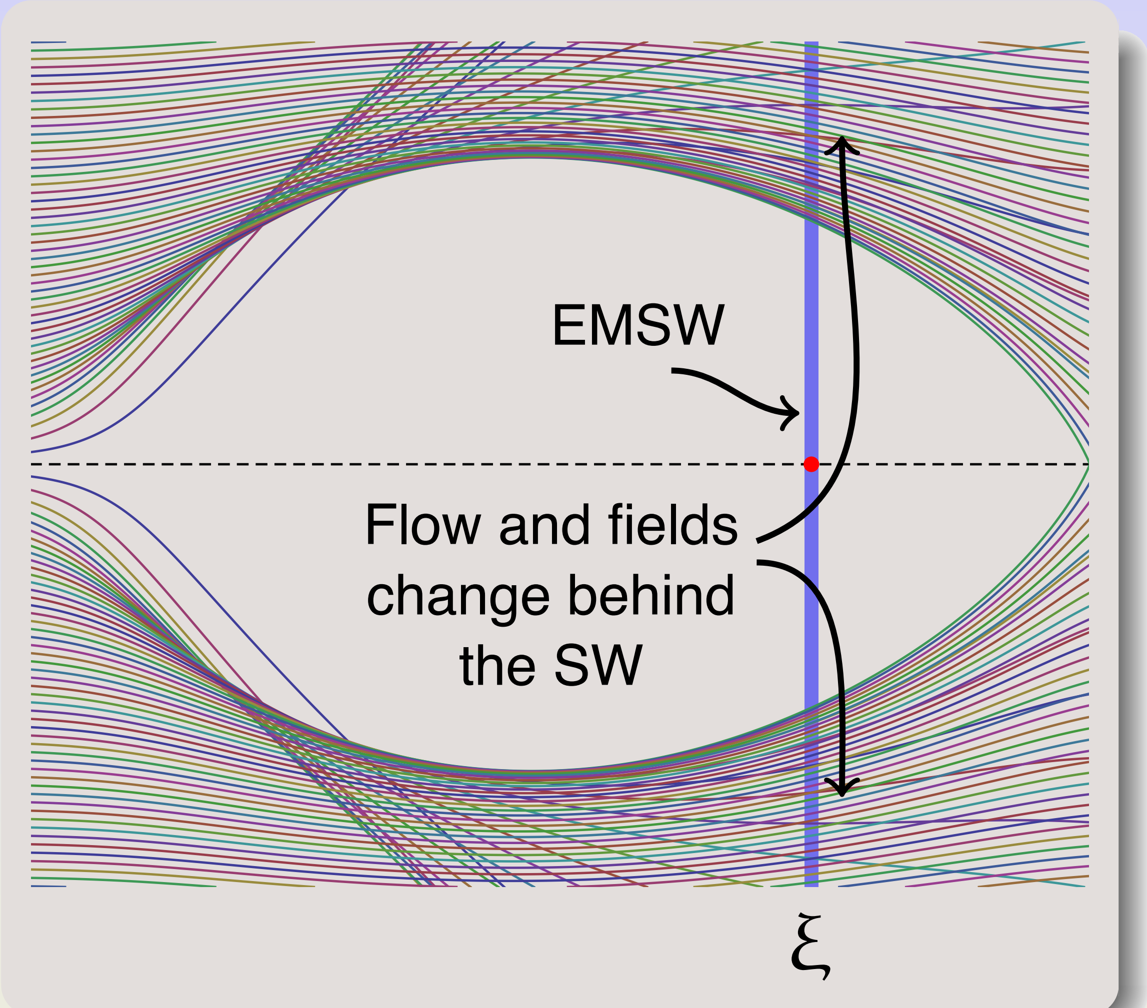
We[6] developed a matlab code that solves an axisymmetric plasma bubble generated by a Gaussian driver and witness bunches. Illustrations: the driver with $\sigma_z = 13$ μm, $\sigma_r = 5$ μm, plasma density 4×10^{16} cm^{-3} ($k_p^{-1} = 26$ μm).

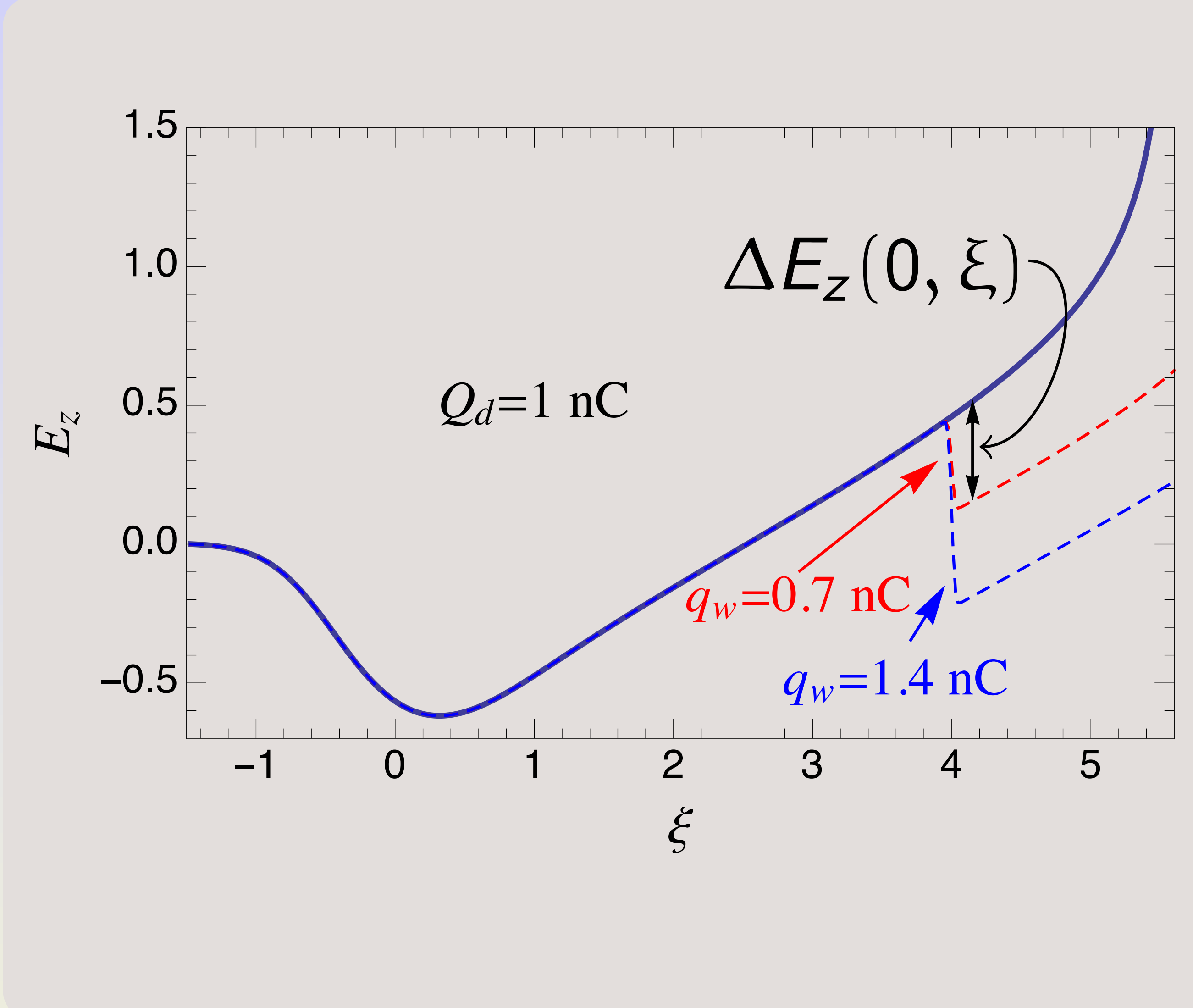


Plots of the longitudinal electric field. One unit of electric field is 19.2 GV/m.

[6] G. Stupakov, P. Baxevanis, V. Khudik, to be published.

Longitudinal wake in the bubble





I developed theory that calculates a jump in E_z immediately behind the witness charge, $\Delta E_z(r, \xi)$. Remarkably, the theory predicts that this jump is proportional to the (dimensionless) witness charge ν_w (the charge has not to be small). So we can introduce the longitudinal wake is $w_\ell = \Delta E_z(0, \xi)/\nu_w$.

Calculation of the longitudinal wake

First, one needs to calculate the strength of the EMSW, $D(r, \xi)$, at the location of the witness charge:

$$E_r(r, \xi) = D(r, \xi_0)\delta(\xi - \xi_0)$$

(here ξ_0 is the position of the source charge in the bubble). It satisfies the following equations

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} rD = \frac{n_{e0}(r, \xi)}{\gamma_0(r, \xi)} D$$

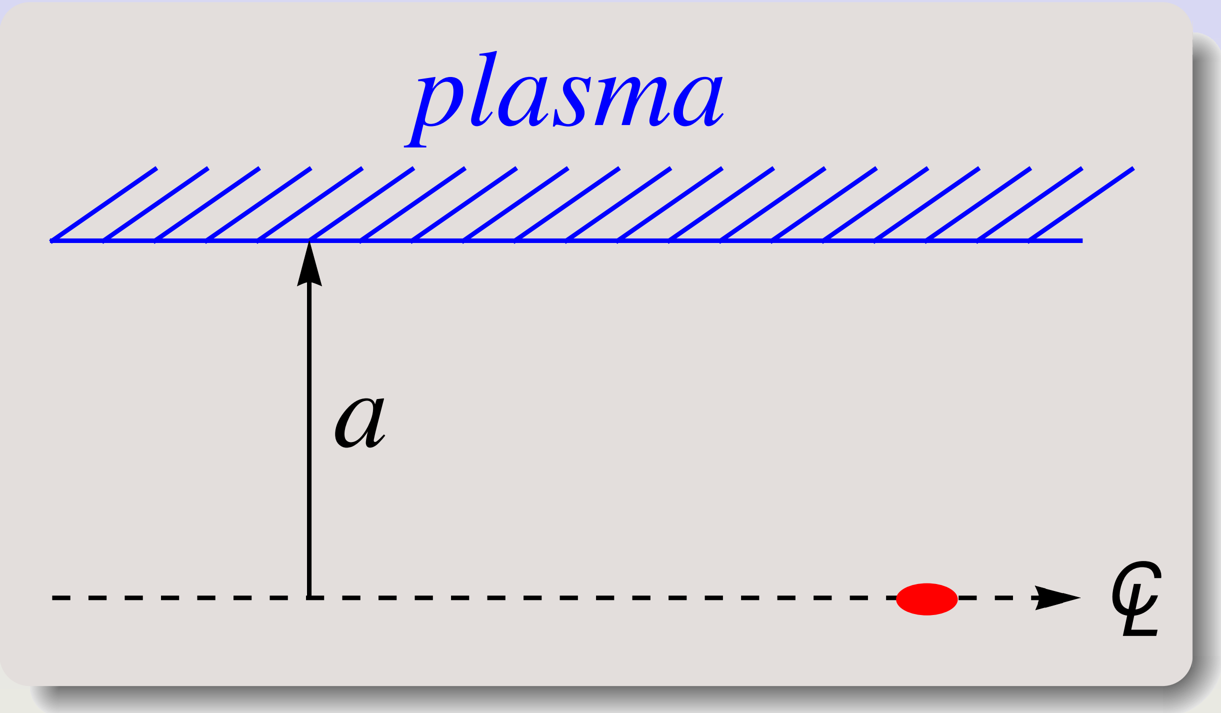
Here n_{e0} and γ_0 are the quantities in the bubble without the witness charge. Then

$$\Delta E_z = -\frac{1}{r}\frac{\partial}{\partial r} rD$$

This result can be benchmarked against the wakefields in a hollow plasma channel.

Wakefields in a hollow plasma channel

Wakefields for a hollow plasma channel were calculated in[7] *in linear approximation* (small charge limit).



Longitudinal wake

$$w_\ell(z) = 2\kappa \cos\left(\frac{\Omega}{c} z\right)$$

$$\kappa = \frac{2}{a^2} \frac{K_0(ak_p)}{K_2(ak_p)}$$

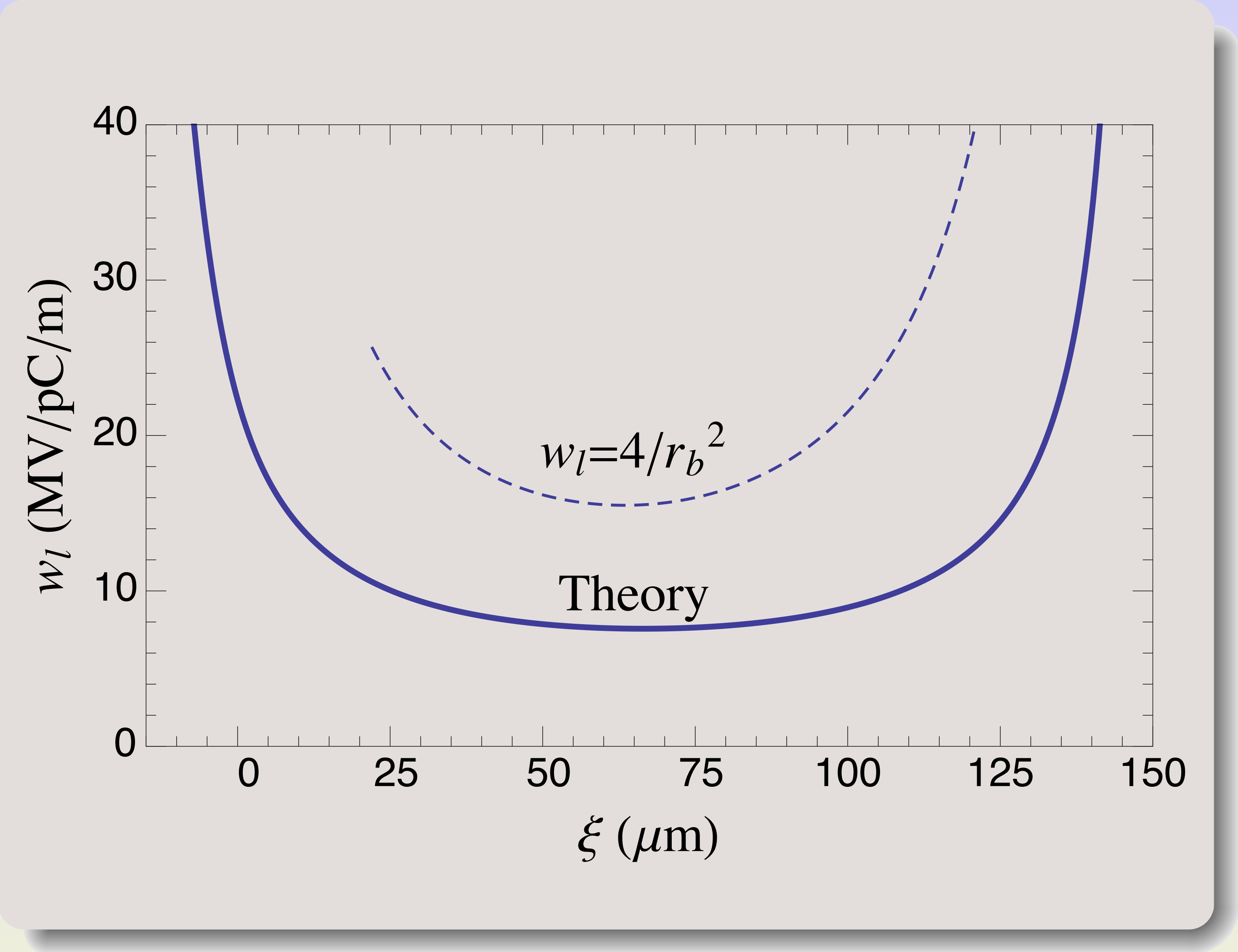
In my analysis I use $n_{e0}(r) = n_0 h(r-a)$ and $\gamma_0(r) = 1$ and obtain

$$w_\ell(0) = \frac{4}{a^2} \frac{K_0(ak_p)}{K_2(ak_p)}$$

The wake $w_\ell(0)$ is valid not only in the linear, but in *nonlinear regime* as well.

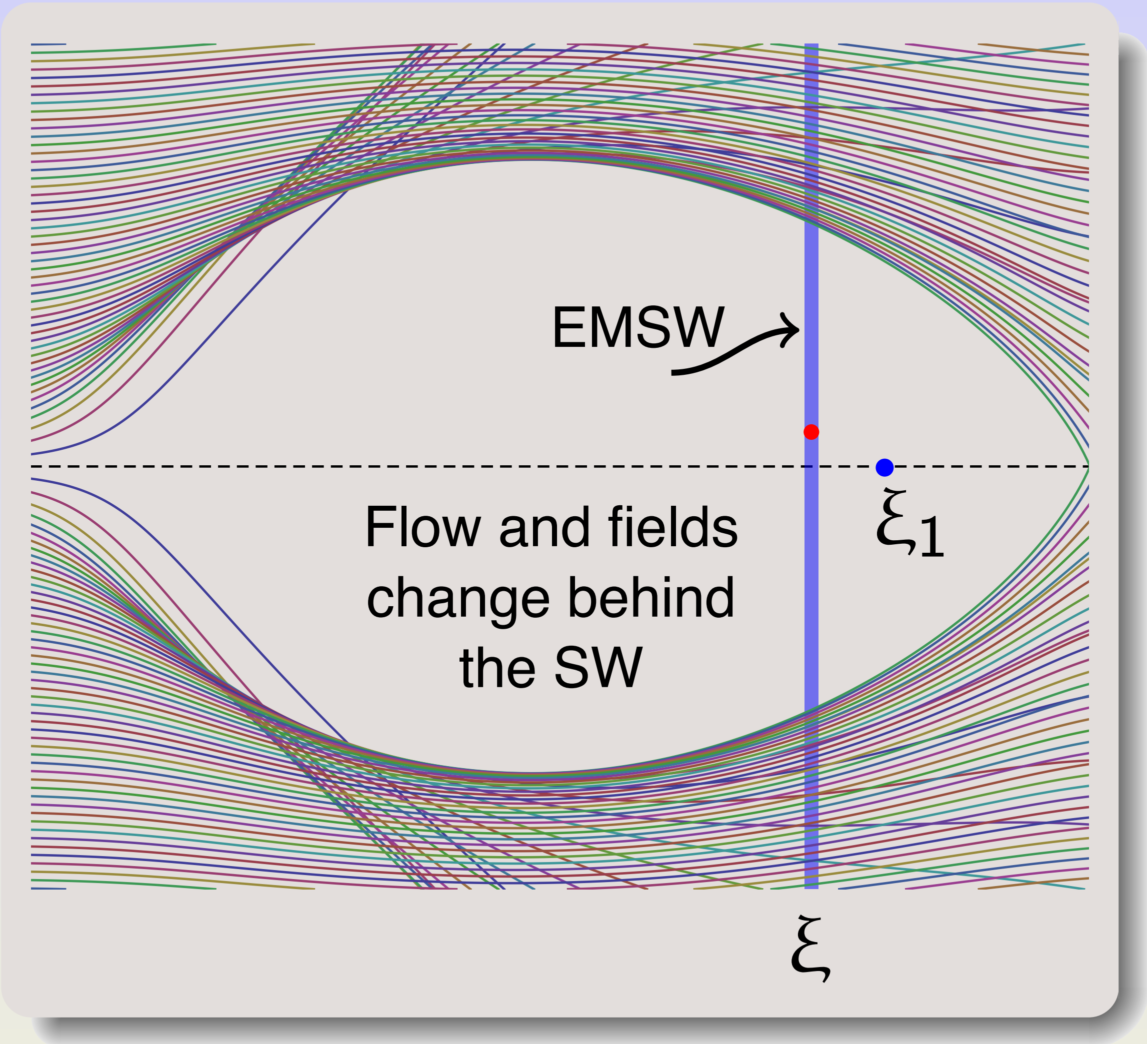
[7] C. Schroeder, D. Whittum, J. Wurtele. PRL, **82**, 1177 (1999).

Longitudinal wake as a function of ξ



This wake is in good agreement with the simulated jump in ΔE_z of a witness charge on the axis of the bubble.

Transverse wake in the bubble



The source charge is now off axis, the offset is assumed small. The shock wave is not axisymmetric, $E_r \propto \hat{D}(r,\xi)\cos\theta$, $E_\theta \propto \hat{D}(r,\xi)\sin\theta$. Behind the wave $\Delta E_z(r,\xi,\theta) = \Delta\hat{E}_z(r,\xi)\cos\theta$. The fields satisfy the following equations

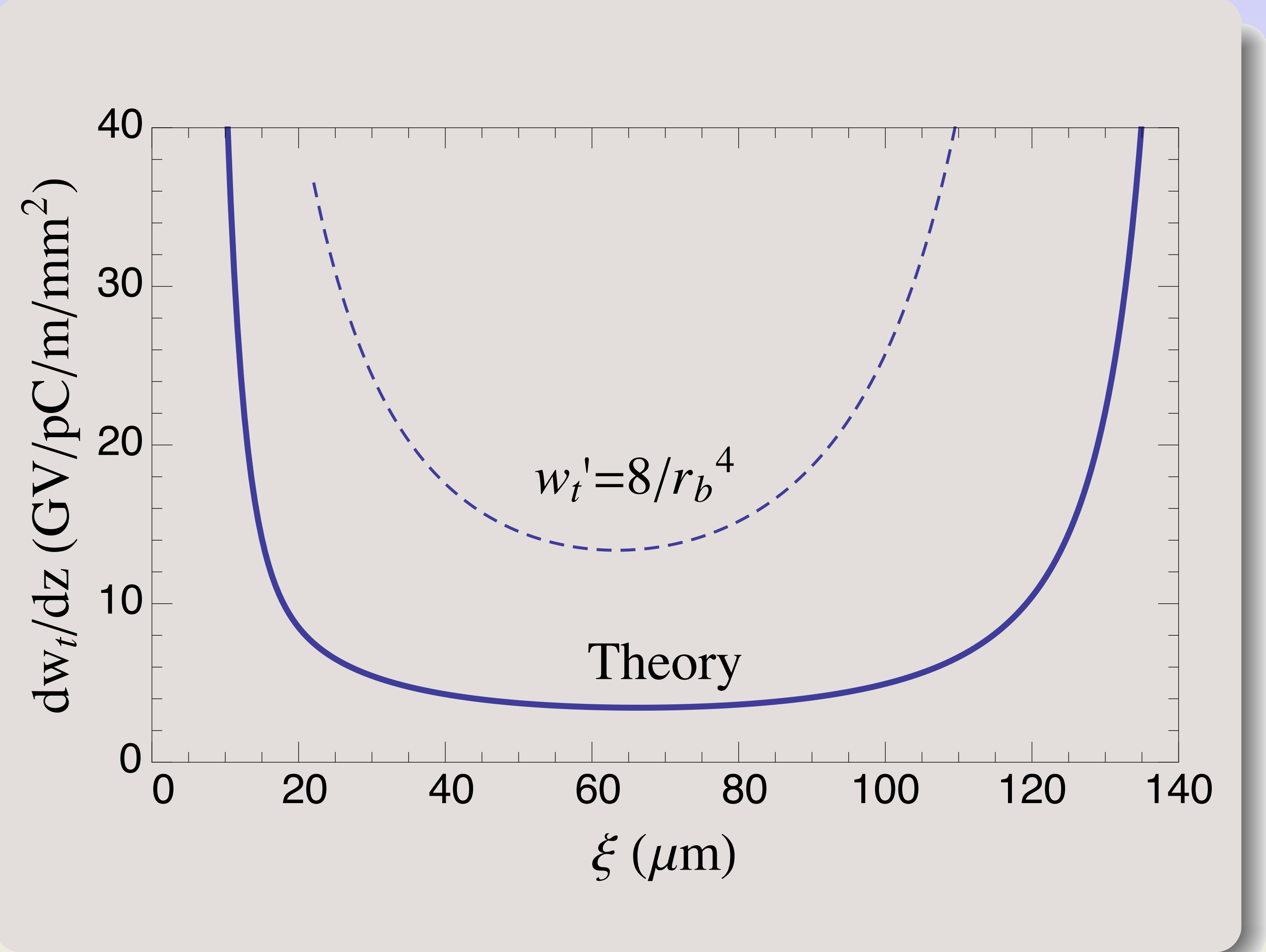
$$\partial_{rr}\hat{D} + \frac{1}{r}\partial_r\hat{D} - \frac{4\hat{D}}{r^2} = \frac{n_{e0}(r,\xi)}{\gamma_0(r,\xi)}\hat{D}$$

$$\Delta\hat{E}_z = -\partial_r\hat{D} - \frac{2\hat{D}}{r}$$

The transverse wake is w_t is found from the Panofsky-Wenzel relation and it is a linear function of the distance between the source and the witness, $w_t = w_t'(\xi_1 - \xi)$. Our result agrees with the linear approximation of the transverse wake in a plasma channel calculated by Schroeder et al.

$$w_t' = \frac{8}{a^4}\frac{K_1(ak_p)}{K_3(ak_p)}$$

Transverse wake as a function of ξ



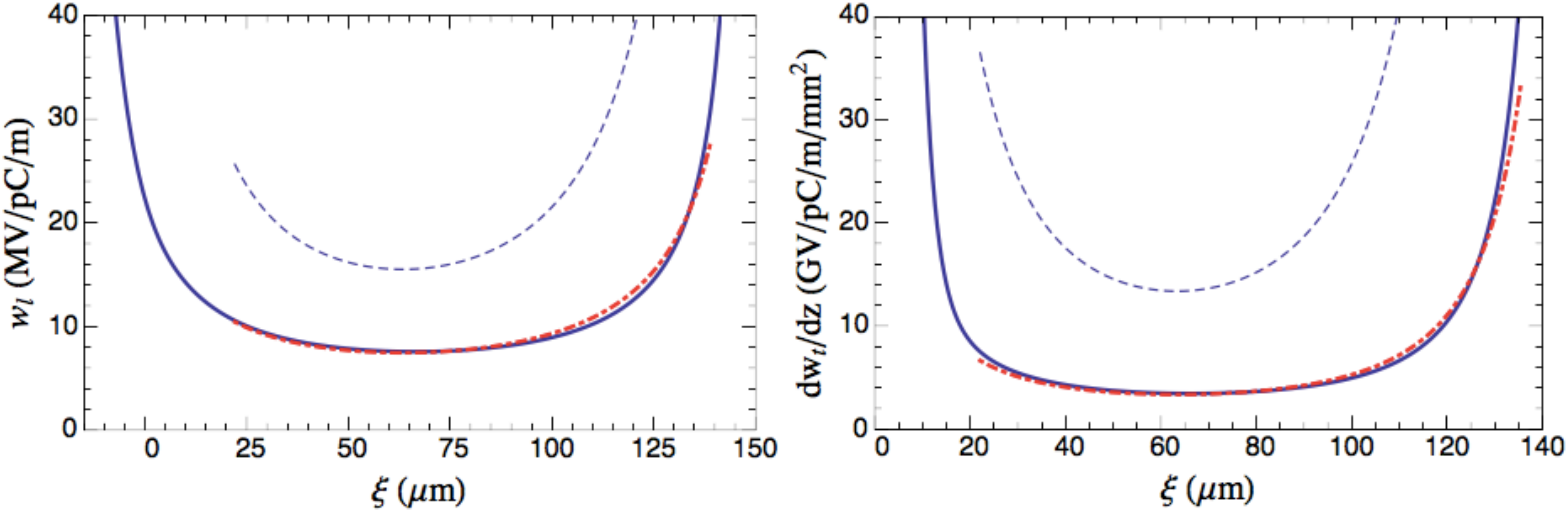


FIG. 4. Longitudinal wake (left panel) and the slope of the transverse wake (right panel) in the plasma bubble shown in Fig. 2. The dashed lines show the wakes calculated using simple formulas for the short-range wakes in a cylindrical pipe. The red dot-dashed lines are plotted using Eqs. (43).

$$w_l(\xi) = \frac{4}{(r_b(\xi) + 0.8k_p^{-1})^2}, \qquad \frac{dw_t}{dz} = \frac{8}{(r_b(\xi) + 0.75k_p^{-1})^4}.$$

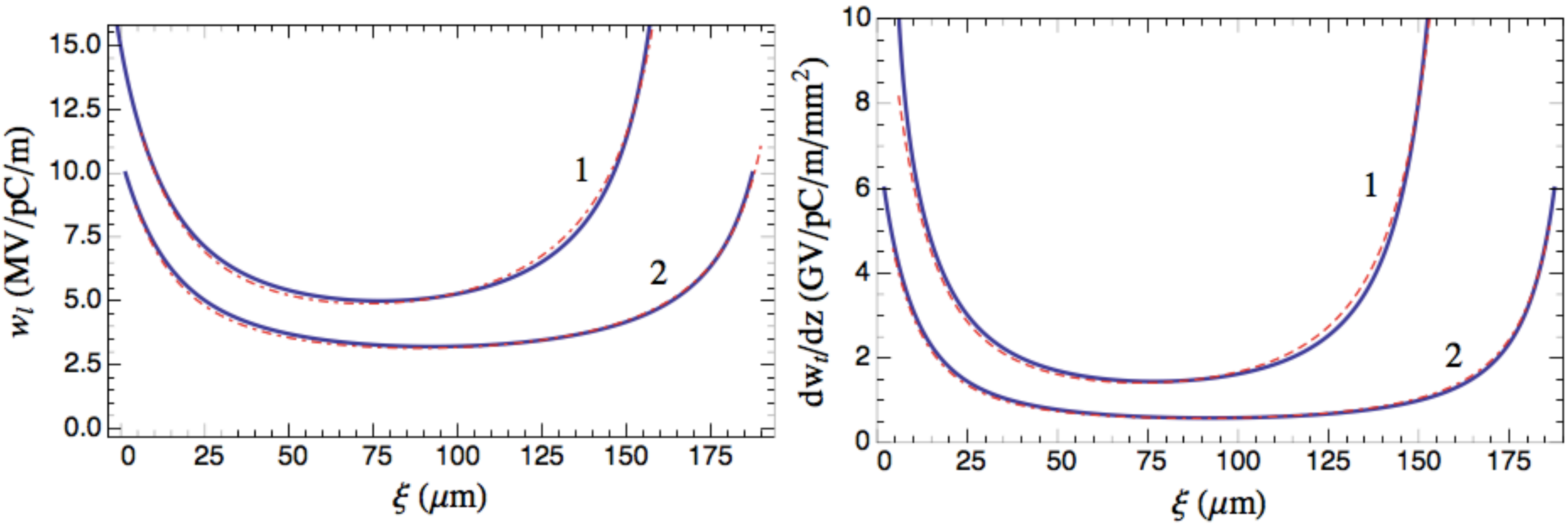
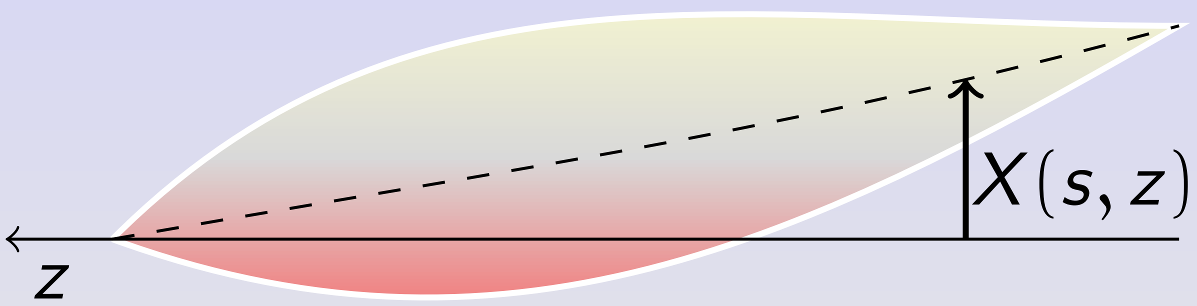


FIG. 5. Longitudinal wake (left panel) and the slope of the transverse wake (right panel) for 2 nC (label 1) and 4 nC (label 2) driver bunches. The red dot-dashed lines are plotted using Eqs. (43).

BBU instability of the witness bunch

With the model for the wakefields in the plasma bubble, we apply them to the beam-breakup instability of the witness bunch.



$X(s,z)$ is the transverse offset of the slice, z is the coordinate in the bunch, s is the distance along the accelerator:

$$\left[\frac{\partial}{\partial s}\gamma(s)\frac{\partial}{\partial s}+\gamma(s)k_\beta^2(s)\right]X(s,z)=N_b r_e\int_\zeta^\infty f_w(z')w_t(z'-z)X(s,z')dz'$$

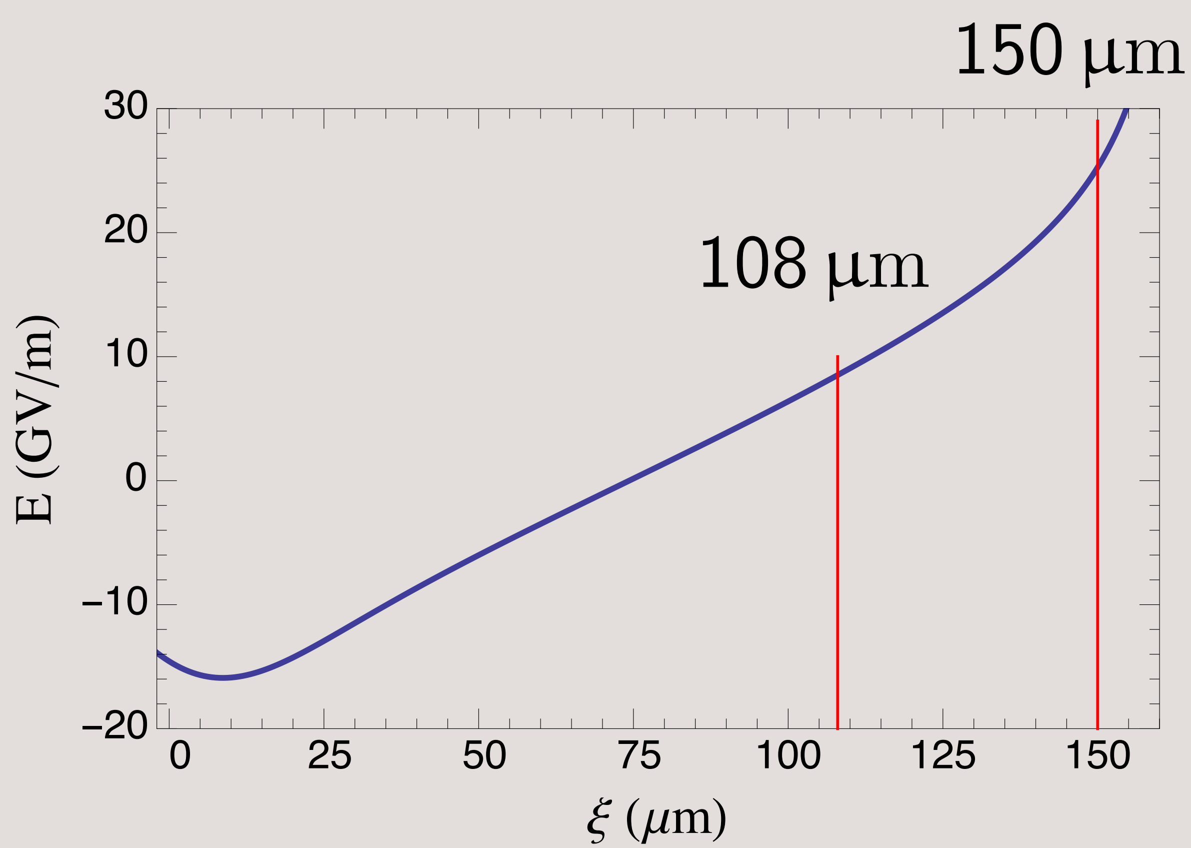
Here $\gamma(s)$ is the energy increase with distance due to acceleration, $k_\beta(s)$ is the focusing, f_w is the longitudinal distribution in the bunch.

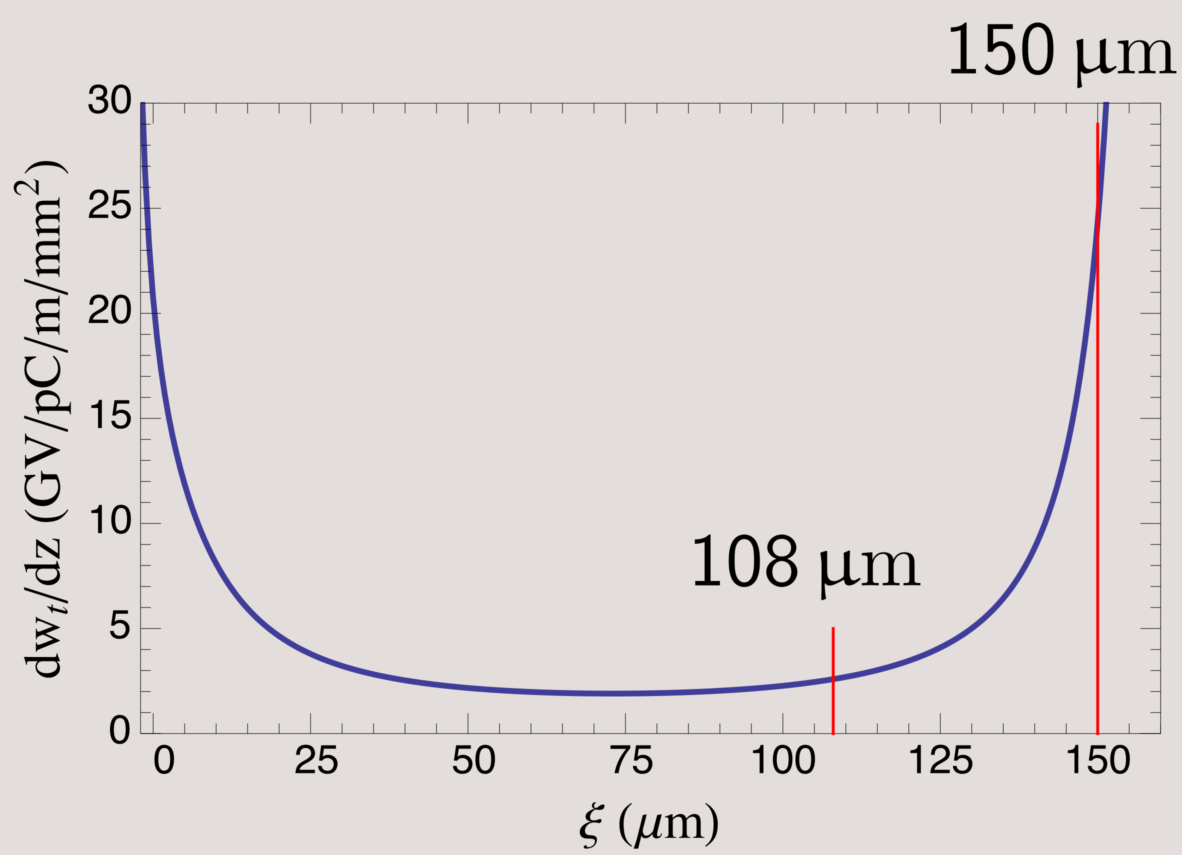
Assume $\gamma(s)=\gamma_0+gs$, $k_\beta(s)=k_0\sqrt{\gamma_0/\gamma(s)}$. If the focusing is due to plasma ions, then $k_0=k_p/\sqrt{2\gamma_0}$.

We can solve the BBU equation numerically for an arbitrary distribution function.

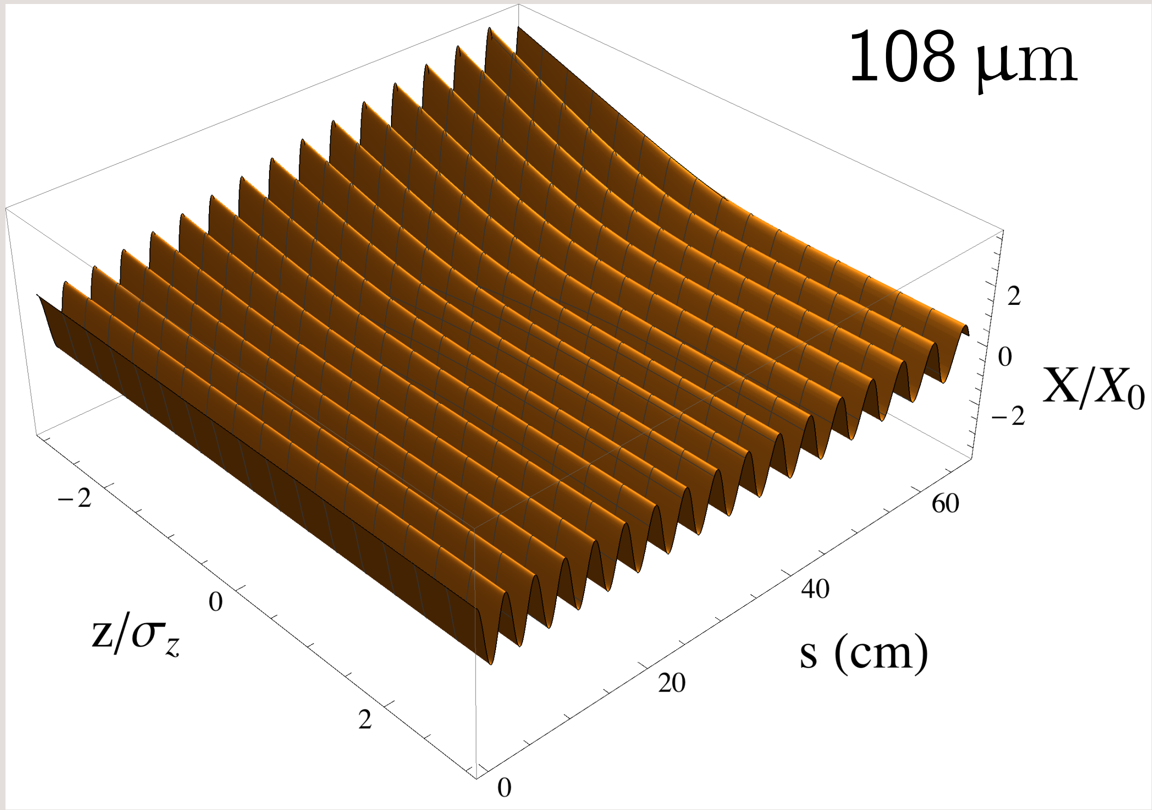
Numerical solution for a Gaussian bunch

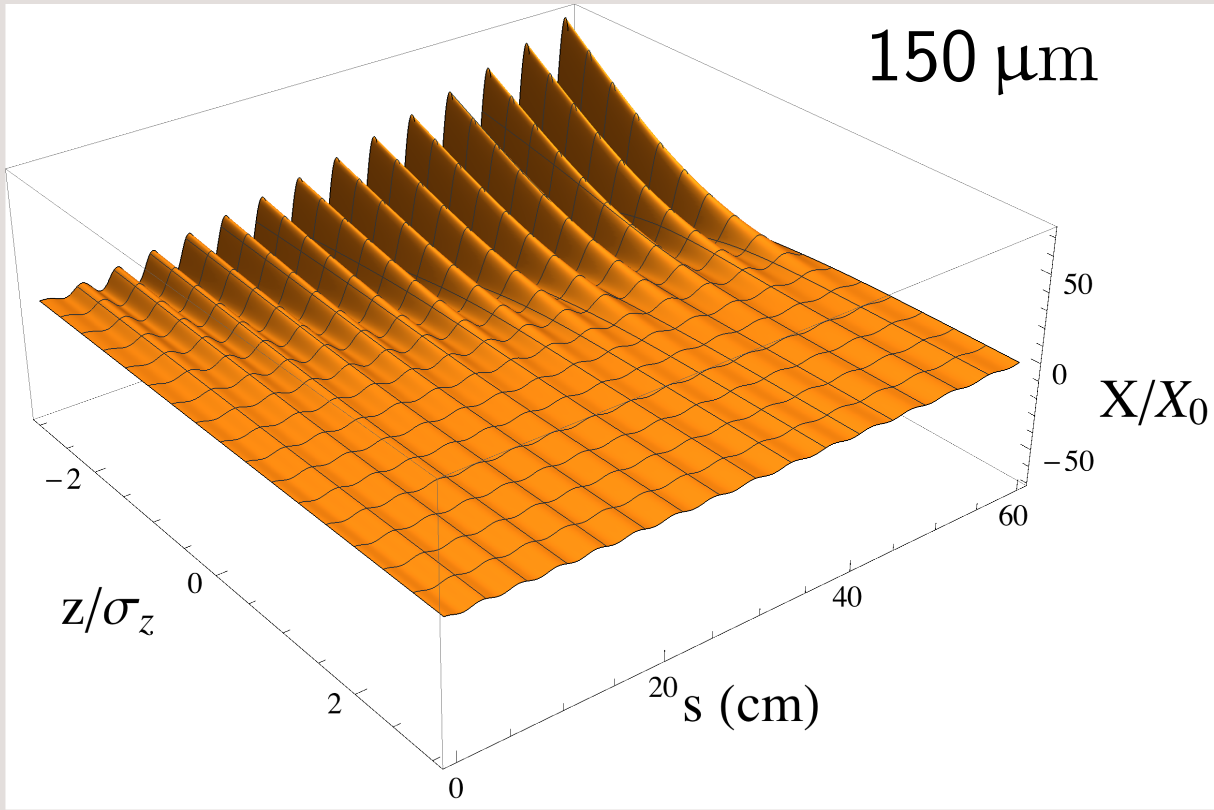
Parameters of Weiming An simulations: the driver has $\sigma_z = 12.77$ µm, $\sigma_r = 3.65$ µm, $Q = 1.6$ nC, ($I_{\rm peak} = 15$ kA); the witness has $\sigma_z = 6.38$ µm, $\sigma_r = 3.65$ µm, $Q = 0.69$ nC, ($I_{\rm peak} = 13$ kA). Plasma density $4 \times 10^{16}\ \mathrm{cm}^{-3}$. The distance between the bunches is a) 108 µm and b) 150 µm.





Numerical solution for a Gaussian witness bunch





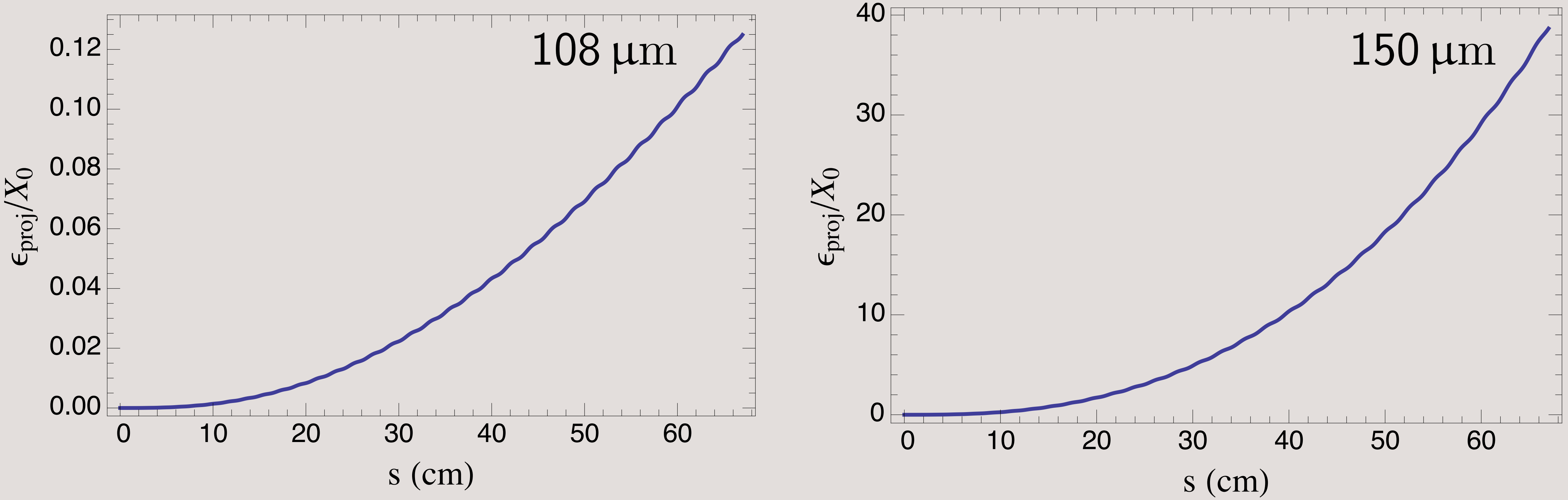
One way to characterize BBU is to calculate the projected emittance:

$$\epsilon^2_{\mathrm{proj}}(s) = \langle (X-\bar{X})^2\rangle\langle (X'-\bar{X}')^2\rangle - \langle (X-\bar{X})(X'-\bar{X}')\rangle$$

where the averaging means

$$\langle \ldots \rangle = \int dz(\ldots) f_w(z)$$

BBU instability—projected emittance



For a particular application this result can be translated into the jitter tolerance for the witness bunch.

Summary

- A method is developed to calculate longitudinal and transverse short-range wakes in the PWFA blowout regime. The calculation requires the knowledge of the energy-density radial distribution in the bubble, which can be taken from 2D simulations of PWFA. We developed a matlab code that solves axisymmetric plasma bubble excited by a driver with arbitrary longitudinal current distribution (run a few minutes on a desktop computer).
- The calculated transverse wakefield is then used for the study of BBU instability. The strength of the instability critically depends on the position of the witness bunch in the bubble.

Acknowledgments

I thank X. Xu for benchmarking numerical calculations of the matlab code and P. Baxevanis for the code development.